ATKINSON SCIENCE

THEORY GUIDE

# Admittance Method

# 3 Composite Wall

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Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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# 1 Complex transmission matrix

In the preceding report in this series, Ref. [1], we introduced the complex transmission matrix for a homogeneous slab with thermal conductivity $k$, thermal diffusivity $\alpha$ and thickness $L$ (see Eqn. (5.3) in Ref. [1]):

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} A_L \\ Q(L) \end{bmatrix} \quad (1.1)$$

where $Q(0)$ and $Q(L)$ are complex constants and

$$M = L\sqrt{j\omega/\alpha}, \quad N = k\sqrt{j\omega/\alpha}$$

We can use (1.1) to find the heat flux in a homogeneous slab when a sinusoidal temperature oscillation about 0°C with amplitude $A_0$ is applied to the face $x = 0$ and the face $x = L$ is held at 0°C. The amplitude $A_L$ must be set to zero in this case.

Alternatively, we can invert the complex transmission matrix, so that we have

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix}^{-1} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix}$$

$$= \begin{bmatrix} \cosh M & -\dfrac{\sinh M}{N} \\ -N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} \quad (1.2)$$

Now we can use (1.2) to find the heat flux in the slab when a sinusoidal temperature oscillation about 0°C with amplitude $A_L$ is applied to the face $x = L$ of the slab and the face $x = 0$ is held at 0°C. The amplitude $A_0$ must be set to zero in this case.

When sinusoidal temperature oscillations are applied to both faces of the slab, we can solve the heat flux for each oscillation separately and add the two solutions together, taking account of any phase difference between the two temperature oscillations.

# 2 Composite wall

Suppose a composite wall consists of $n$ slabs of different materials in parallel. The response of the wall can be modelled by recognising that the heat output from one slab becomes the input to the next slab. Hence, the complex transmission matrices of the slabs can be multiplied in sequence to provide an overall complex transmission matrix.

For a composite wall, (1.1) becomes

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} A_L \\ Q(L) \end{bmatrix} \qquad (2.1)$$

where

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} \cosh M_1 & \dfrac{\sinh M_1}{N_1} \\ N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \begin{bmatrix} \cosh M_2 & \dfrac{\sinh M_2}{N_2} \\ N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh M_{n-1} & \dfrac{\sinh M_{n-1}}{N_{n-1}} \\ N_{n-1} \sinh M_{n-1} & \cosh M_{n-1} \end{bmatrix} \begin{bmatrix} \cosh M_n & \dfrac{\sinh M_n}{N_n} \\ N_n \sinh M_n & \cosh M_n \end{bmatrix} \qquad (2.2)$$

The thickness $L$ of the composite slab is

$$L = \sum_{i=1}^{n} l_i$$

where $l_i$ is the thickness of the $i$th slab. We must calculate $M_1$, $M_2$, …, $M_{n-1}$, $M_n$ and $N_1$, $N_2$, …, $N_{n-1}$, $N_n$ before we can calculate the elements in the matrices and carry out the matrix multiplication.

We can multiply the inverse transmission matrices of the slabs in sequence to obtain the overall inverse transmission matrix. Eq. (1.2) becomes:

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} \qquad (2.3)$$

where

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} \cosh M_n & -\dfrac{\sinh M_n}{N_n} \\ -N_n \sinh M_n & \cosh M_n \end{bmatrix} \begin{bmatrix} \cosh M_{n-1} & -\dfrac{\sinh M_{n-1}}{N_n} \\ -N_n \sinh M_n & \cosh M_n \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh M_2 & -\dfrac{\sinh M_2}{N_2} \\ -N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \begin{bmatrix} \cosh M_1 & -\dfrac{\sinh M_1}{N_1} \\ -N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \qquad (2.4)$$

To obtain (2.4) we have used the rule $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, where $\mathbf{A}$ and $\mathbf{B}$ are square matrices. The rule enables us to write $(\mathbf{ABC})^{-1} = (\mathbf{BC})^{-1}\mathbf{A}^{-1} = (\mathbf{C}^{-1}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$, and so on for greater numbers of matrices. As before, we must calculate $M_1$, $M_2$, …, $M_{n-1}$, $M_n$ and $N_1$, $N_2$, …, $N_{n-1}$, $N_n$ before we can calculate the elements in the matrices and carry out the matrix multiplication.

# 3 Example 1

A wall is made up of the materials shown in Table 1. Calculate (a) the complex transmission matrix of the wall, and (b) the inverse complex transmission matrix.

**Table 1 Materials in the composite wall**

| $i$ | Material | Thickness [mm] | Density [kg m$^{-3}$] | Specific heat capacity [J kg$^{-1}$ K$^{-1}$] | Thermal conductivity [W m$^{-1}$ K$^{-1}$] |
|---|---|---|---|---|---|
| 1 | Solid brick | 220 | 1750 | 1000 | 0.77 |
| 2 | Mineral wool | 50 | 12 | 1030 | 0.042 |
| 3 | Plasterboard | 12.5 | 700 | 1000 | 0.21 |

(a) The complex transmission matrix for an individual material $i$ is

$$\begin{bmatrix} \cosh M_i & \dfrac{\sinh M_i}{N_i} \\ N_i \sinh M_i & \cosh M_i \end{bmatrix}$$

where the complex constants $M_i$ and $N_i$ are

$$M_i = l_i\sqrt{j\omega/\alpha_i} \quad \text{and} \quad N_i = k_i\sqrt{j\omega/\alpha_i}$$

The thermal diffusivity $\alpha_i$ [m$^2$ s$^{-1}$] is

$$\alpha_i = \frac{k_i}{\rho_i C_i}$$

For diurnal temperature variations $\omega = 2\pi \div (60 \times 60 \times 24) = 2\pi/86400$ rad s$^{-1}$.

For the outer solid brick layer, the thermal diffusivity $\alpha_1$ is

$$\alpha_1 = \frac{k_1}{\rho_1 C_1} = \frac{0.77}{1750 \times 1000} = 0.44 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$$

The complex constant $M_1$ is therefore

$$M_1 = l_1\sqrt{j\omega/\alpha_1} = 0.22 \times \sqrt{j2\pi/(86400 \times 0.44 \times 10^{-6})}$$

$$= 2.8283256\sqrt{j} = 2.8283256\frac{1}{\sqrt{2}}(1+j) = 1.9999282(1+j)$$

and the complex constant $N_1$ is

$$N_1 = k_1\sqrt{j\omega/\alpha_1} = 0.77\sqrt{j2\pi/(86400 \times 0.44 \times 10^{-6})}$$

$$= 9.8991396\sqrt{j} = 9.8991396\frac{1}{\sqrt{2}}(1+j) = 6.9997488(1+j)$$

Table 2 gives $\alpha_i$, $M_i$ and $N_i$ for all of the materials.

**Table 2 $\alpha_i$, $M_i$ and $N_i$ for individual materials**

| $i$ | Material | $\alpha_i$ [m² s⁻¹] | $M_i$ | $N_i$ |
|---|---|---|---|---|
| 1 | Solid brick | $0.44 \times 10^{-6}$ | $1.9999282\ (1 + j)$ | $6.9997488\ (1 + j)$ |
| 2 | Mineral wool | $3.3980583 \times 10^{-6}$ | $0.1635583\ (1 + j)$ | $0.1373890\ (1 + j)$ |
| 3 | Plasterboard | $0.3 \times 10^{-6}$ | $0.1376155\ (1 + j)$ | $2.3119409\ (1 + j)$ |

For the brick outer layer, cosh $M_1$ is

$$\cosh M_1 = \cosh[1.9999282(1 + j)]$$

$$= \frac{1}{2}\left[e^{1.9999282(1+j)} + e^{-1.9999282(1+j)}\right]$$

$$= \frac{1}{2}\left[e^{1.9999282} e^{j1.9999282} + e^{-1.9999282} e^{-j1.9999282\times10^{-3}}\right]$$

$$= 3.6942628e^{j1.9999282} + 0.0676725e^{-j1.9999282}$$

$$= 3.6942628 \times [\cos 1.9999282 + j \sin 1.9999282]$$

$$+ 0.0676725 \times [\cos 1.9999282 - j \sin 1.9999282]$$

$$= -1.5652692 + j3.2977588$$

and sinh $M_1$ is

$$\sinh M_1 = \sinh[1.9999282(1 + j)]$$

$$= \frac{1}{2}\left[e^{1.9999282(1+j)} - e^{-1.9999282(1+j)}\right]$$

$$= \frac{1}{2}\left[e^{1.9999282} e^{j1.9999282} - e^{-1.9999282} e^{-j1.9999282}\right]$$

$$= 3.6942628e^{j1.9999282} - 0.0676725e^{-j1.9999282}$$

$$= 3.6942628 \times [\cos 1.9999282 + j \sin 1.9999282]$$

$$-0.0676725 \times [\cos 1.9999282 - j \sin 1.9999282]$$

$$= -1.5089547 + j3.4208317$$

and so

$$\frac{\sinh M_1}{N_1} = \frac{-1.5089547 + j3.4208317}{6.9997488(1+j)}$$

$$= \frac{(1-j)(-0.21557269 + j0.48870778)}{(1-j)(1+j)}$$

$$= \frac{-0.21557269 + j0.48870778 + j0.21557269 - j^2 0.48870778}{1-j^2}$$

$$= 0.13656755 + j0.35214024$$

and

$$N_1 \sinh M_1 = 6.9997488(1+j)(-1.5089547 + j3.4208317)$$

$$= 6.9997488(-1.5089547 + j3.4208317 - j1.5089547 + j^2 3.4208317)$$

$$6.9997488(-4.9297864 + j1.911877)$$

$$= -34.507266 + j13.382659$$

The complex transmission matrix for the solid brick is therefore

<u>Solid brick</u>

$$\boldsymbol{A}_1 = \begin{bmatrix} \cosh M_1 & \dfrac{\sinh M_1}{N_1} \\ N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} = \begin{bmatrix} (-1.5652692 + j3.2977588) & (0.13656755 + j0.35214024) \\ (-34.507266 + j13.382659) & (-1.5652692 + j3.2977588) \end{bmatrix}$$

The complex transmission matrix for the mineral wool is

<u>Mineral wool</u>

$$\boldsymbol{A}_2 = \begin{bmatrix} \cosh M_2 & \dfrac{\sinh M_2}{N_2} \\ N_2 \sinh M_2 & \cosh M_2 \end{bmatrix}$$

$$= \begin{bmatrix} (0.9998807 + j0.0267511) & (1.1904476 + j0.0106156) \\ (-0.000400753 + j0.0449412) & (0.9998807 + j0.0267511) \end{bmatrix}$$

and the complex transmission matrix for the plasterboard is

<u>Plasterboard</u>

$$\boldsymbol{A}_3 = \begin{bmatrix} \cosh M_3 & \dfrac{\sinh M_3}{N_3} \\ N_3 \sinh M_3 & \cosh M_3 \end{bmatrix}$$

$$= \begin{bmatrix} (0.9999403 + j0.0189379) & (0.0595231 + j0.000375754) \\ (-0.00401686 + j0.6363102) & (0.9999403 + j0.0189379) \end{bmatrix}$$

We obtain the complex transmission matrix for the wall by multiplying the individual matrices together:

$$\boldsymbol{z} = \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3$$

The multiplication of matrices satisfies the associative law, so

$$\boldsymbol{z} = \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3 = \boldsymbol{A}_1(\boldsymbol{A}_2\boldsymbol{A}_3) = (\boldsymbol{A}_1\boldsymbol{A}_2)\boldsymbol{A}_3$$

Multiplying $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ gives

$$\boldsymbol{A}_1\boldsymbol{A}_2 = \begin{bmatrix} (-1.66918 + j3.26149) & (-1.77125 + j4.26494) \\ (-35.0087 + j12.3863) & (-42.8745 + j18.8205) \end{bmatrix}$$

and multiplying $\boldsymbol{A}_1\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ gives

$$\boldsymbol{z} = (\boldsymbol{A}_1\boldsymbol{A}_2)\boldsymbol{A}_3 = \begin{bmatrix} (-4.43756 + j2.08549) & (-1.95249 + j4.42465) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix}$$

(b) The inverse of the complex transmission matrix is

$$\boldsymbol{Z} = \begin{bmatrix} (-45.3168 + j18.7315) & (1.95249 - j4.42465) \\ (47.0447 + j15.6345) & (-4.43756 + j2.08549) \end{bmatrix}$$

# 4 Heat fluxes

## 4.1 Temperature variation on the outside surface, $x = 0$

For a composite wall, the heat transmission when the outside surface, $x = 0$, is subjected to a sinusoidal temperature variation about 0°C with amplitude $A_0$°C is given by (2.1) with $A_L$ set to zero:

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q(L) \end{bmatrix}$$

where $Q(0)$ and $Q(L)$ are complex constants. The matrix equation gives

$$Q(0) = A_0 \frac{z_4}{z_2}$$

and

$$Q(L) = A_0 \frac{1}{z_2}$$

The sinusoidal temperature variation on the outside surface, $x = 0$, is

$$\theta(0, t) = A_0 \sin(\omega t + \phi) = \mathrm{Im}\left[A_0 e^{j(\omega t + \phi)}\right] \qquad (4.1)$$

For diurnal temperature variations, the angular speed $\omega$ is $2\pi \div (60 \times 60 \times 24) = 2\pi/86400$ rad s$^{-1}$. If the peak in temperature occurs at 15:00, then the offset $\phi$ must be set to $-2\pi(15 - 6)/24 = -0.75\pi$ rad.

At the outside surface, $x = 0$, the variation in heat flux is

$$q(0, t) = \mathrm{Im}\left[Q(0) e^{j(\omega t + \phi)}\right]$$

$$= \mathrm{Im}\left[A_0 \frac{z_4}{z_2} e^{j(\omega t + \phi)}\right] \qquad (4.2)$$

and at the inside surface, $x = L$, the variation in heat flux is

$$q(L, t) = \mathrm{Im}\left[Q(L) e^{j(\omega t + \phi)}\right]$$

$$= \mathrm{Im}\left[A_0 \frac{1}{z_2} e^{j(\omega t + \phi)}\right] \qquad (4.3)$$

## 4.2 Temperature variation on the inside surface, $x = L$

For a composite wall, the heat transmission when the inside surface, $x = L$, is subjected to a sinusoidal temperature variation about 0°C with amplitude $A_L$°C is given by (2.3) with $A_0$ set to zero:

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q(0) \end{bmatrix}$$

where $Q(0)$ and $Q(L)$ are complex constants. The matrix equation gives

$$Q(0) = A_L \frac{1}{Z_2}$$

and

$$Q(L) = A_L \frac{Z_4}{Z_2}$$

The sinusoidal temperature variation on the inside surface, $x = L$, is

$$\theta(L, t) = A_L \sin(\omega t + \varphi) = \operatorname{Im}\left[A_L e^{j(\omega t + \varphi)}\right] \quad (4.4)$$

If the peak in temperature occurs at 12:00, then the offset $\varphi$ must be $-2\pi(12 - 6)/24 = -0.5\pi$ rad.

At the outside surface, $x = 0$, the variation in heat flux is

$$q(0, t) = \operatorname{Im}\left[Q(0) e^{j(\omega t + \varphi)}\right]$$

$$= \operatorname{Im}\left[A_L \frac{1}{Z_2} e^{j(\omega t + \varphi)}\right] \quad (4.5)$$

and at the inside surface, $x = L$, the variation in heat flux is

$$q(L, t) = \operatorname{Im}\left[Q(L) e^{j(\omega t + \varphi)}\right]$$

$$= \operatorname{Im}\left[A_L \frac{Z_4}{Z_2} e^{j(\omega t + \varphi)}\right] \quad (4.6)$$

# 5 Example 2

The composite wall in Example 1 is subjected to a daily sinusoidal temperature variation on its outside surface having a mean of 15°C and an amplitude of 10°C. The peak in temperature occurs at 3:00 pm. The inside surface is subjected to a daily sinusoidal temperature variation having a mean of 20°C and an amplitude of 5°C. The peak temperature occurs at 12:00 noon. Calculate:

a. the mean heat flux through the wall,
b. the heat fluxes through the inside and outside surfaces due to the temperature variation on the outside surface,
c. the heat fluxes through the inside and outside surfaces due to the temperature variation on the inside surface, and
d. the net heat fluxes on the inside and outside surfaces.

(a) We must calculate the steady thermal transmittance ($U$ value) of the composite wall before we can calculate the mean heat flux. The steady thermal transmittance of a composite wall with $n$ layers is given by

$$\frac{1}{U} = \sum_{i=1}^{n} \frac{l_i}{k_i}$$

where $l_i$ and $k_i$ are the thickness and the thermal conductivity of the $i^{\text{th}}$ layer, respectively. Hence

$$\frac{1}{U} = \frac{0.22}{0.77} + \frac{0.05}{0.042} + \frac{0.0125}{0.21} = 1.5357143 \text{ m}^2 \text{ K W}^{-1}$$

so $U$ is 0.6511628 W m$^{-2}$ K$^{-1}$. The mean heat flux through the wall $\bar{q}$ [W m$^{-2}$] is

$$\bar{q} = -U(\bar{\theta}_L - \bar{\theta}_0)$$

where

$$\bar{\theta}_0 = 15°\text{C} \qquad (5.1)$$

and

$$\bar{\theta}_L = 20°\text{C} \qquad (5.2)$$

The mean heat flux is therefore

$$\bar{q} = -0.6511628 \times (20 - 15) = -3.255814 \text{ W m}^{-2} \qquad (5.3)$$

The negative sign indicates that the heat is flowing from the inside to the outside.

(b) From Example 1, the complex transmission matrix for the composite wall is

$$\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} (-4.43756 + j2.08549) & (-1.95249 + j4.42465) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix}$$

The sinusoidal temperature variation on the outside surface, $x = 0$, is given by (4.1):

$$\theta(0,t) = A_0 \sin(\omega t + \phi) = \mathrm{Im}\left[A_0 e^{j(\omega t + \phi)}\right]$$

For diurnal temperature variations, the angular speed $\omega$ is $2\pi \div (60 \times 60 \times 24) = 2\pi/86400$ rad s$^{-1}$. The peak in temperature occurs at 15:00, so the offset $\phi$ must be $-2\pi(15-6)/24 = -0.75\pi$ rad. The amplitude $A_0$ of the temperature variation is 10°C, so (4.1) becomes

$$\theta(0,t) = 10\sin(\omega t - 0.75\pi) = \mathrm{Im}\left[10 e^{j(\omega t - 0.75\pi)}\right] \quad (5.4)$$

Substituting the values of $A_0$ and $\phi$ into (4.2) gives the variation in heat flux at the outside surface, $x = 0$:

$$q(0,t) = \mathrm{Im}\left[A_0 \frac{z_4}{z_2} e^{j(\omega t + \phi)}\right] = \mathrm{Im}\left[10 \frac{z_4}{z_2} e^{j(\omega t - 0.75\pi)}\right] \quad (5.5)$$

and substituting the values of $A_0$ and $\phi$ into (4.3) gives the variation in heat flux at the inside surface, $x = L$:

$$q(L,t) = \mathrm{Im}\left[A_0 \frac{1}{z_2} e^{j(\omega t + \phi)}\right] = \mathrm{Im}\left[10 \frac{1}{z_2} e^{j(\omega t - 0.75\pi)}\right] \quad (5.6)$$

The term $1/z_2$ required in (5.6) is

$$\frac{1}{z_2} = \frac{1}{-1.95249 + j4.42465}$$

$$= \frac{-1.95249 - j4.42465}{(-1.95249 + j4.42465)(-1.95249 - j4.42465)}$$

$$= \frac{-1.95249 - j4.42465}{1.95249^2 + 4.42465^2}$$

$$= -0.083476328 - j0.18917051$$

The complex number $1/z_2$ can be represented in the complex plane as shown in Figure 1. The amplitude of $1/z_2$ is

$$\text{Amplitude} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{(-0.08348)^2 + (-0.18917)^2} = 0.20677$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the inside surface of the wall will lag the temperature variation on the outside surface, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase in the clockwise (negative) direction from the positive Real axis gives

$$\text{Phase} = -1.9864 \text{ rad } (= -113.8°)$$

We can now write $1/z_2$ as

$$\frac{1}{z_2} = 0.20677[\cos(-1.9864) + j\sin(-1.9864)]$$

$$= 0.20677e^{-j1.9864} \qquad (5.7)$$

Substituting (5.7) into (5.6) gives the heat flux at the inside surface, $x = L$:

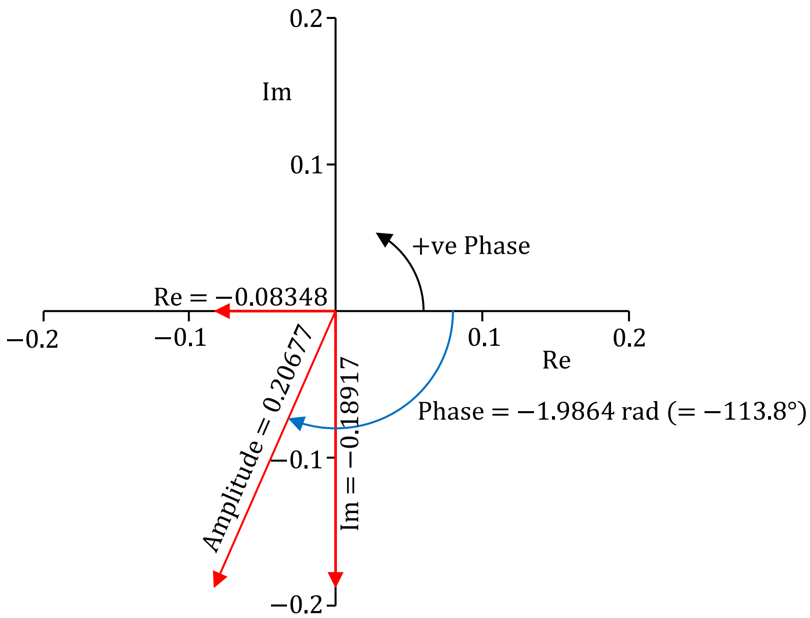
$$q(L,t) = \mathrm{Im}\left[10 \times 0.20677e^{-j1.9864}e^{j(\omega t - 0.75\pi)}\right]$$

$$= \mathrm{Im}\left[2.0677e^{j(\omega t - 0.75\pi - 1.9864)}\right]$$

$$= 2.0677\sin(\omega t - 0.75\pi - 1.9864) \qquad (5.8)$$

The peak heat flux at $x = L$ lags the peak temperature at $x = 0$ by 1.9864 rad (= 113.8°). In terms of hours, the lag is 24 hr × 113.8°/360° = 7.587 hr (7 hr 35 min).

**Figure 1 Amplitude and phase of $1/z_2$**



The term $z_4/z_2$ required in (5.5) is

$$\frac{z_4}{z_2} = (-45.3168 + j18.7316)(-0.083476328 - j0.18917051)$$

$$= 3.7828801 + j8.5726022 - j1.5636452 - j^2 3.5434663$$

$$= 7.3263464 + j7.0089570$$

The complex number $z_4/z_2$ can be represented in the complex plane as shown in Figure 2. The amplitude of $z_4/z_2$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{7.3263^2 + 7.00896^2} = 10.1390$$

We know the heat flux variation on the outside surface of the wall will lead the temperature variation on the outside surface, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase in the anticlockwise (positive) direction from the positive Real axis gives

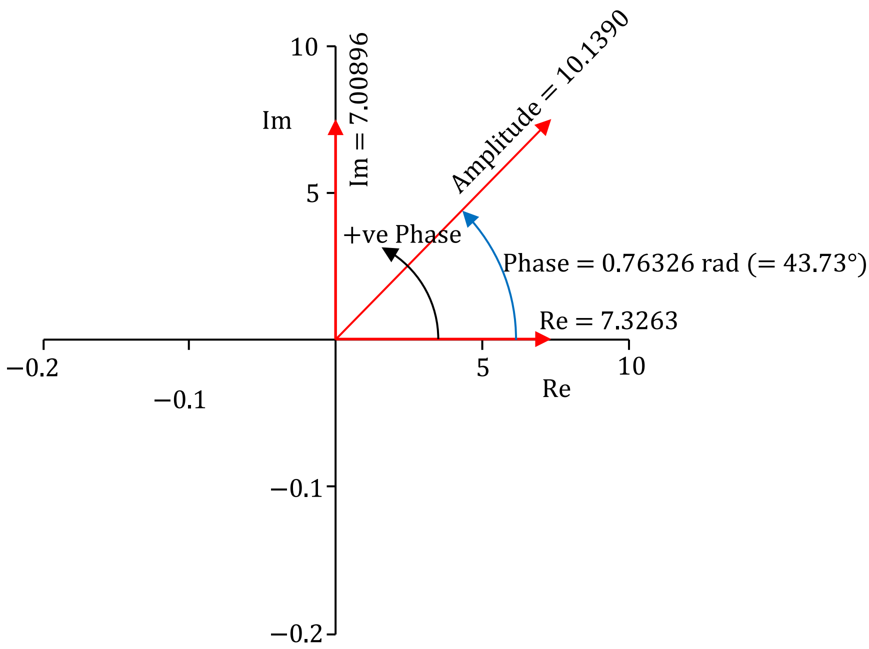
$$\text{Phase} = \ 0.76326 \text{ rad } (= 43.73°)$$

We can now write $z_4/z_2$ as

$$\frac{z_4}{z_2} = 10.1390(\cos 0.76326 + j \sin 0.76326)$$

$$= 10.1390 e^{j0.76326} \qquad (5.9)$$

**Figure 2 Amplitude and phase of $z_4/z_2$**



Substituting (5.9) into (5.5) gives

$$q(0,t) = \mathrm{Im}\left[10 \times 10.1390 e^{j0.76326} e^{j(\omega t - 0.75\pi)}\right]$$

$$= \mathrm{Im}\left[101.390 e^{j(\omega t - 0.75\pi + 0.76326)}\right]$$

$$= 101.390 \sin(\omega t - 0.75\pi + 0.76326) \qquad (5.10)$$

The peak heat flux at $x = 0$ leads the peak temperature at $x = 0$ by $0.76326$ rad ($= 43.73°$). In terms of hours, the lead is $24$ hr $\times$ $43.73°/360° = 2.915$ hr (2 hr 55 min).

(c) From Example 1, the inverse of the complex transmission matrix for the composite wall is

$$\boldsymbol{Z} = \begin{bmatrix} (-45.3168 + j18.7315) & (1.95249 - j4.42465) \\ (47.0447 + j15.6345) & (-4.43756 + j2.08549) \end{bmatrix}$$

The sinusoidal temperature variation on the inside surface, $x = L$, is given by (4.4):

$$\theta(L, t) = A_L \sin(\omega t + \varphi) = \mathrm{Im}\left[A_L e^{j(\omega t + \varphi)}\right]$$

The peak in temperature occurs at 12:00, so the offset $\varphi$ must be $-2\pi(12 - 6)/24 = -0.5\pi$ rad. The amplitude $A_L$ of the temperature variation is 5°C, so (4.4) becomes

$$\theta(L, t) = 5 \sin(\omega t - 0.5\pi) = \mathrm{Im}\left[5e^{j(\omega t - 0.5\pi)}\right] \quad (5.11)$$

Substituting the values of $A_L$ and $\varphi$ into (4.5) gives the variation in heat flux at the outside surface, $x = 0$:

$$q(0, t) = \mathrm{Im}\left[5\frac{1}{Z_2}e^{j(\omega t - 0.5\pi)}\right] \quad (5.12)$$

and substituting the values of $A_L$ and $\varphi$ into (4.6) gives the variation in heat flux at the inside surface, $x = L$:

$$q(L, t) = \mathrm{Im}\left[5\frac{Z_4}{Z_2}e^{j(\omega t - 0.5\pi)}\right] \quad (5.13)$$

The term $1/Z_2$ required in (5.12) is

$$\frac{1}{Z_2} = \frac{1}{1.95249 - j4.42465}$$

$$= \frac{1.95249 + j4.42465}{(1.95249 - j4.42465)(1.95249 + j4.42465)}$$

$$= \frac{1.95249 + j4.42465}{1.95249^2 + 4.42465^2}$$

$$= 0.08348 + j0.18917$$

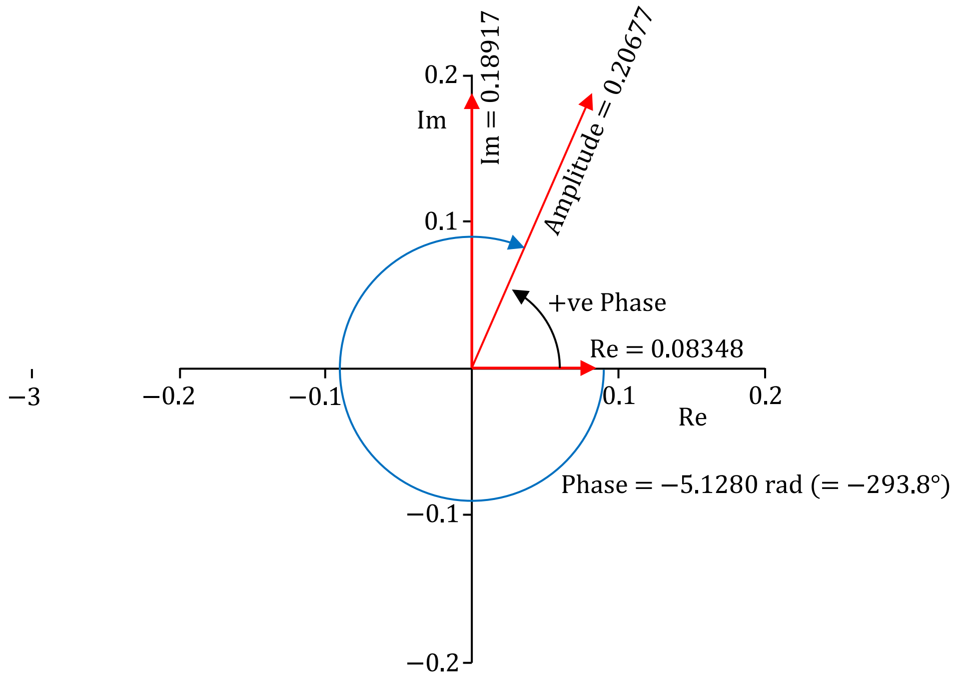
The complex number $1/Z_2$ can be represented in the complex plane as shown in Figure 3. The amplitude of $1/Z_2$ is

$$\text{Amplitude} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{0.08348^2 + 0.18917^2} = 0.20677$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the outside surface of the wall will lag the temperature variation on the inside surface, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase in the clockwise (negative) direction from the positive Real axis gives

$$\text{Phase} = -5.1280 \text{ rad } (= -293.8°)$$

**Figure 3 Amplitude and phase of $1/Z_2$**




We can now write $1/Z_2$ as

$$\frac{1}{Z_2} = 0.20677[\cos(-5.1280) + j\sin(-5.1280)]$$

$$= 0.20677e^{-j5.1280} \qquad (5.14)$$

Substituting (5.14) into (5.12) gives

$$q(0,t) = \mathrm{Im}\left[5 \times 0.20677e^{-j5.1280}e^{j(\omega t - 0.5\pi)}\right]$$

$$= \mathrm{Im}\left[1.0338e^{j(\omega t - 0.5\pi - 5.1280)}\right]$$

$$= 1.0338\sin(\omega t - 0.5\pi - 5.1280) \qquad (5.15)$$

The peak heat flux of $+1.0338$ W m$^{-2}$ at $x = 0$ lags the peak temperature at $x = L$ by 5.1280 rad (= 293.8°). In terms of hours, the lag is 24 hr × 293.8°/360° = 19.587 hr (19 hr 35 min). This is the time difference between the positive peak in $\theta(L, t)$ and the positive peak in $q(0, t)$. We would expect a positive peak in temperature at $x = L$ to give rise to a *negative* peak in heat flux at $x = 0$, because heat flows in the negative $x$ direction when $\theta(L, t)$ is positive. The time lag between the negative peak in heat flux and the positive peak in temperature is 19 hr 35 min − 12 hr = 7 hr 35 min, which is much shorter.

Notice that 7 hr 35 min is the same as the time between the temperature oscillation $\theta(0, t)$ applied to the outside surface $x = 0$ and the corresponding heat flux on the inside surface $x = L$. The amplitude 0.20677 is also the same.

The term $Z_4/Z_2$ required in (5.13) is

$$\frac{Z_4}{Z_2} = (-4.43756 + j2.08549)(0.08348 + j0.18917)$$

$$= -0.3704475 - j0.8394532 + j0.1740967 + j^2 0.3945121$$

$$= -0.7649596 - j0.6653565$$

The complex number $Z_4/Z_2$ can be represented in the complex plane as shown in Figure 4. The amplitude of $Z_4/Z_2$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(-0.76496)^2 + (-0.66536)^2} = 1.0138355$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the inside surface of the wall will lead the temperature variation on the inside surface, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase in the anticlockwise (positive) direction from the positive Real axis gives

$$\text{Phase} = 4.6762 \text{ rad } (= 267.93°)$$

We can now write $Z_4/Z_2$ as

$$\frac{Z_4}{Z_2} = 1.01384[\cos 3.8575 + j \sin 3.8575]$$

$$= 1.01384 e^{j3.8575} \qquad (5.16)$$

Substituting (5.16) into (5.13) gives

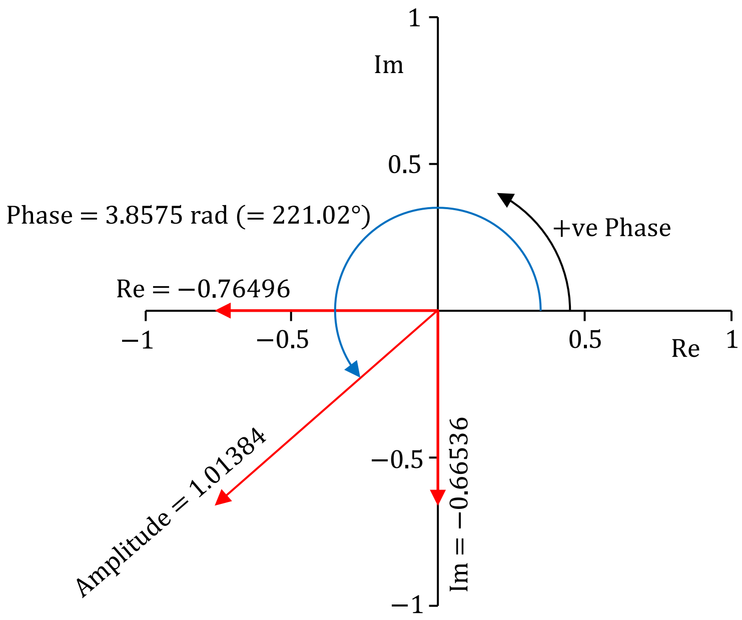
$$q(L,t) = \text{Im}\left[5 \times 1.01384 e^{j3.8575} e^{j(\omega t - 0.5\pi)}\right]$$

$$= \text{Im}\left[5.0692 e^{j(\omega t - 0.5\pi + 3.8575)}\right]$$

$$= 5.0692 \sin(\omega t - 0.5\pi + 3.8575) \qquad (5.17)$$

The peak heat flux of $+5.0692$ W m$^{-2}$ at $x = L$ leads the peak temperature at $x = L$ by 3.8575 rad ($= 221.02°$). In terms of hours, the lead is 24 hr × $221.02°/360°$ = 14.735 hr (14 hr 44 min). This is the time difference between the positive peak in $\theta(L,t)$ and the positive peak in $q(L,t)$. We would expect a positive peak in temperature at $x = L$ to be caused by a *negative* peak in heat flux at $x = L$. The time lead between the negative peak in heat flux and the positive peak in temperature is 14 hr 44 min $-$ 12 hr = 2 hr 44 min, which is much shorter.

**Figure 4  Amplitude and phase of $Z_4/Z_2$**



(d) We can use the superposition principle to determine the net heat fluxes on the inside and outside surfaces of the wall. The process is illustrated in Figure 5.

**Figure 5 Heat fluxes at $x = 0$ and $x = L$**

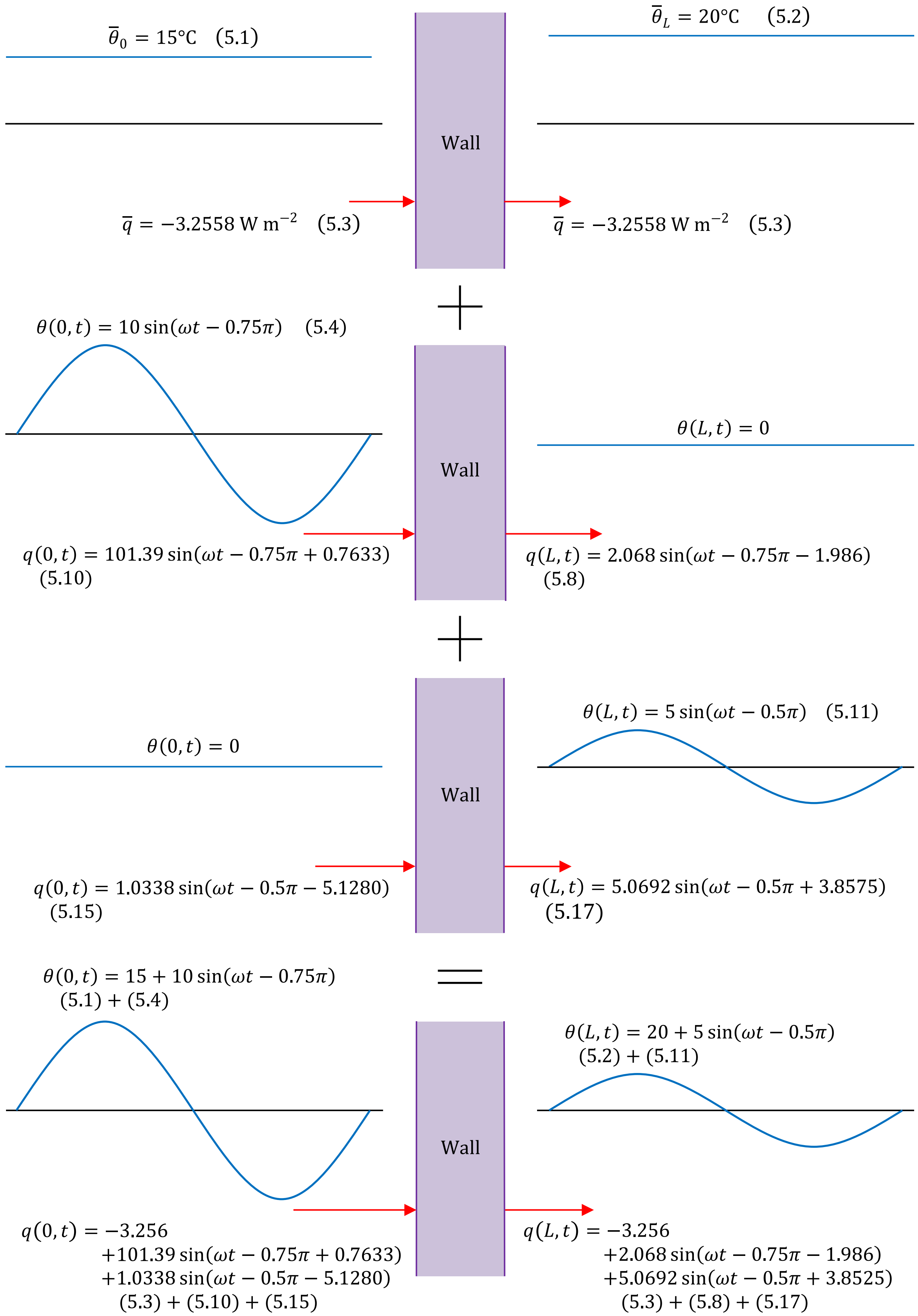


Figure 6 shows the fluctuating components of the temperature on the outside and inside surfaces. On the outside surface the fluctuating temperature reaches a peak at 15:00 hr. On the inside surface the fluctuating temperature reaches a peak at 12:00 hr.

**Figure 6 Fluctuating temperatures on the outside and inside surfaces**

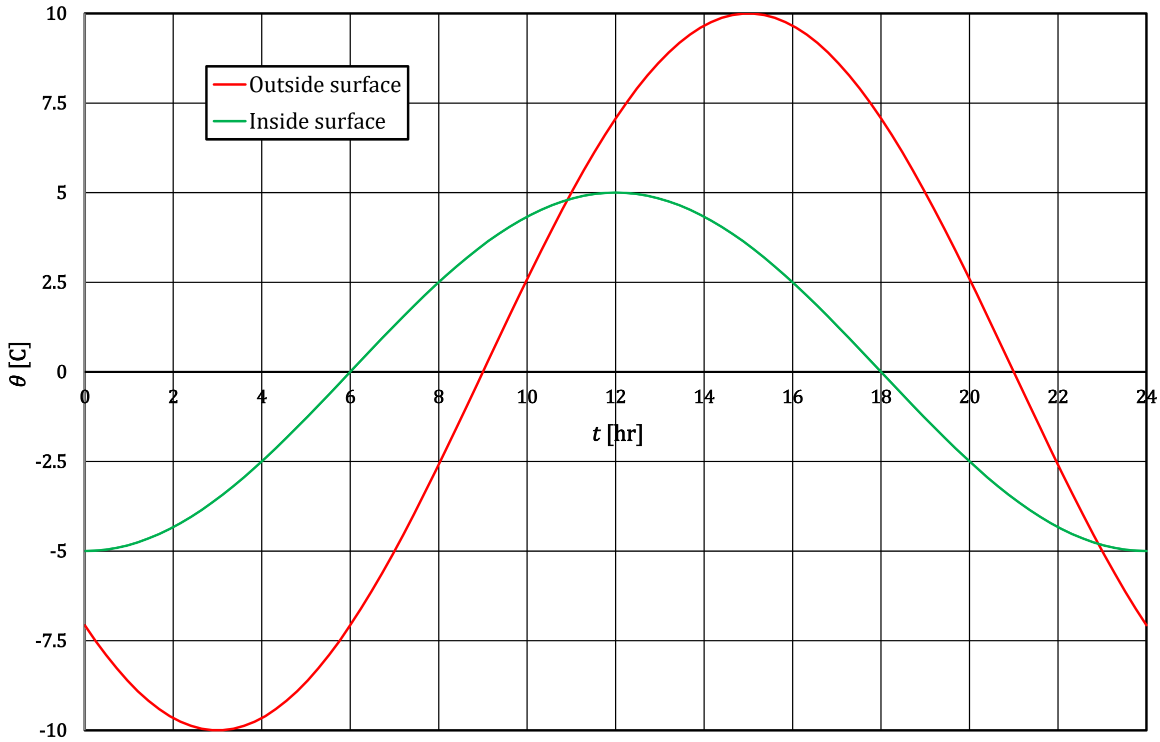


Figure 7 shows the heat fluxes through the outside surface, $x = 0$, and the net heat flux. The heat flux $q$ (5.3) is due to the difference in mean temperature between the inside and outside surfaces. The heat flux $q$ (5.10) is due to the temperature fluctuation on the outside surface $\theta(0, t)$. This heat flux *leads* $\theta(0, t)$ by 2 hr 55 min. The heat flux $q$ (5.15) is due to the temperature fluctuation on the inside surface $\theta(L, t)$. This heat flux *lags* $\theta(L, t)$ by 7 hr 35 min. The amplitude of the heat flux $q$ (5.15) is very small because the wall is well insulated. When added together the three terms give the net heat flux through the outside surface.

**Figure 7 Heat fluxes for the outside surface**

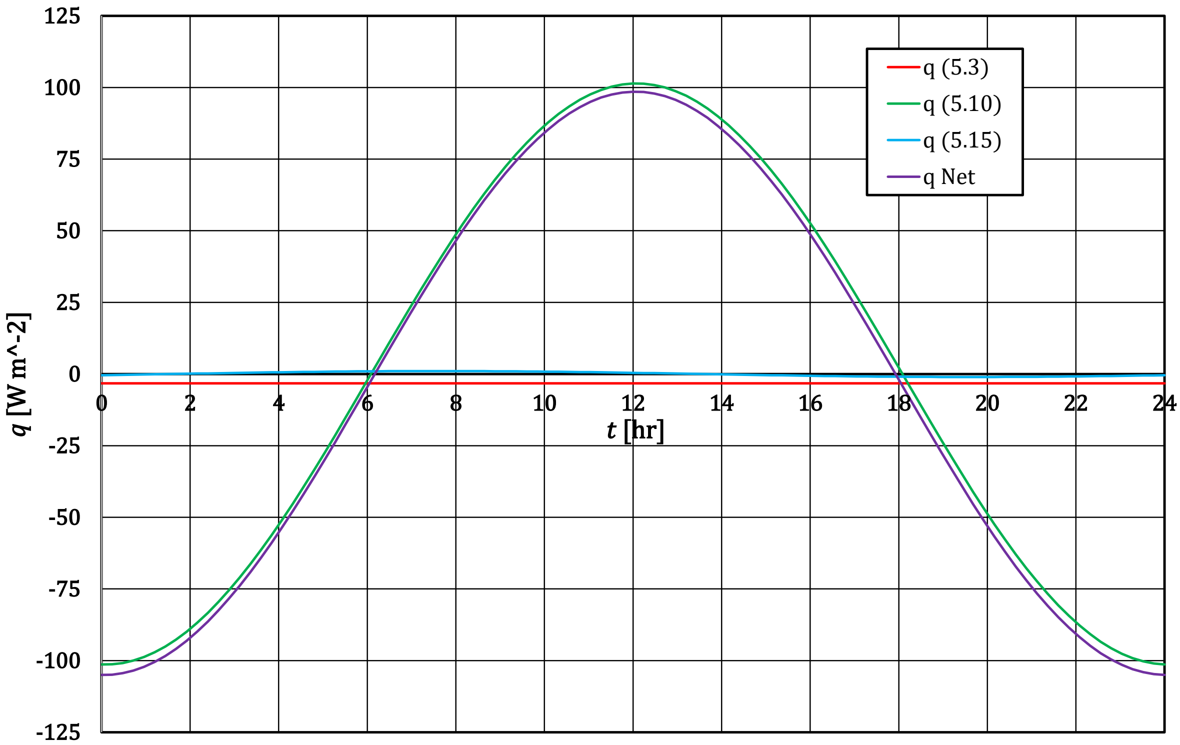
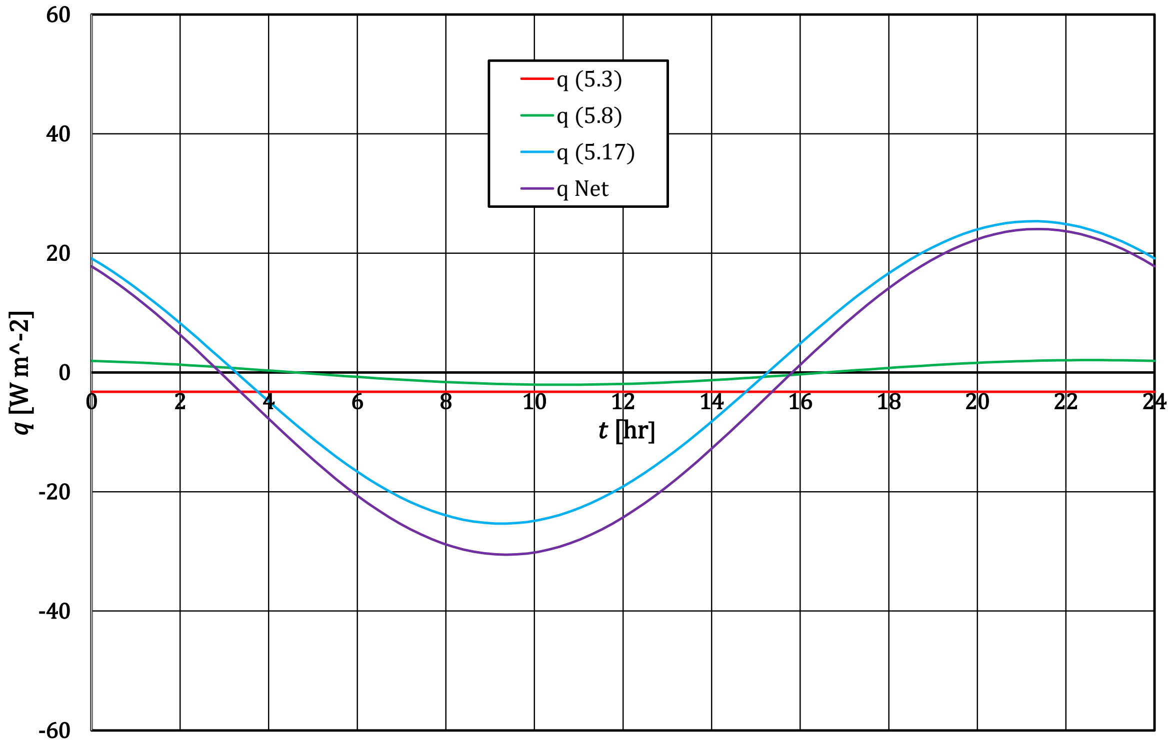


Figure 8 shows the heat fluxes through the inside surface, $x = L$, and the net heat flux. The heat flux $q$ (5.3) is due to the difference in mean temperature between the inside and outside surfaces. The heat flux $q$ (5.8) is due to the temperature fluctuation on the outside surface $\theta(0, t)$. This heat flux lags $\theta(0, t)$ by 7 hr 35 min. The amplitude of $q$ (5.8) is very small because the wall is well insulated. The heat flux $q$ (5.17) is due to the temperature fluctuation on the inside surface $\theta(L, t)$. This heat flux leads $\theta(L, t)$ by 2 hr 44 min. When added together the three terms give the net heat flux through the inside surface.

**Figure 8 Heat fluxes for the inside surface**



# 6 References

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